Calculus and Numeric Methods Formulas

I. Common Greek Letters

Lowercase

α	alpha	ε, ϵ	epsilon	ι	iota	ν	nu	ρ, ϱ	rho	φ, ϕ	phi
β	beta	ζ	zeta	κ	kappa	ξ	xi	σ, ς	sigma	χ	chi
γ	gamma	η	eta	λ	lambda	o	omicron	τ	tau	ψ	psi
δ	delta	θ, ϑ	theta	μ	mu	π, ϖ	pi	υ	upsilon	ω	omega

Capitals

Γ	Gamma	Π	Pi
Δ	Delta	Σ	Sigma
Θ	Theta	Φ	Phi
Λ	Lambda	Ω	Omega

II. Algebra

II. A - Remarkable identities (valid in $\mathbb{C}$, so in $\mathbb{R}$)

$(a+b)^2 = a^2 + 2ab + b^2$; $(a-b)^2 = a^2 - 2ab + b^2$

$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

$(a+b)^n = a^n + C_n^1 a^{n-1} b + \cdots + C_n^k a^{n-k} b^k + \cdots + b^n$, where $C_n^k = n!/(k!\,(n-k)!)$

$a^2 - b^2 = (a+b)(a-b)$; $a^2 + b^2 = (a+ib)(a-ib)$

II. B – Quadratic formula

Let a, b, c be three real numbers with $a \neq 0$, and $\Delta = b^2 - 4ac$
The equation $ax^2 + bx + c = 0$ has:

- if $\Delta > 0$, two real solutions $x_1 = \frac{-b+\sqrt{\Delta}}{2a}$ and $x_2 = \frac{-b-\sqrt{\Delta}}{2a}$
- if $\Delta = 0$, one real solution $x_1 = x_2 = -\frac{b}{2a}$
- if $\Delta < 0$, two complex solutions $x_1 = \frac{-b+i\sqrt{-\Delta}}{2a}$ and $x_2 = \frac{-b-i\sqrt{-\Delta}}{2a}$

In all cases: $ax^2 + bx + c = 0 = a(x - x_1)(x - x_2)$; $x_1 + x_2 = -\frac{b}{a}$; $x_1 x_2 = \frac{c}{a}$

II. C – Arithmetic progression

Arithmetic series: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

Geometric series: $1 + b + b^2 + \cdots + b^n = \frac{1-b^{n+1}}{1-b}$ (if $b \neq 1$)

Factorial: $1 \times 2 \times \cdots \times (n-1) \times n = n!$ (with n positive integer and $0! = 1$ by definition)

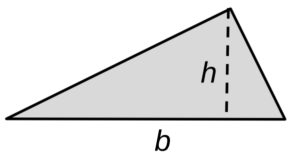
III. Geometry

Equations of simple structures:
Line through $(0, b)$ with slope *a* : $y = ax + b$
Circle with center (a, b) and radius *r* : $(x-a)^2 + (y-b)^2 = r^2$

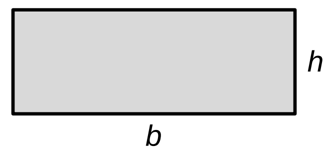
Pythagorean theorem:
In a right triangle with edges *a* and *b* and hypotenuse *c* : $a^2 + b^2 = c^2$

Areas and volumes:

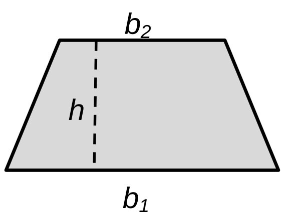
Triangle area
$A = \frac{1}{2} bh$
Tetrahedron volume
$V = \frac{1}{3} Ah$



Rectangle area
$A = bh$

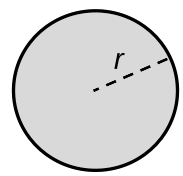


Trapezoid area
$A = \frac{b_1 + b_2}{2} h$



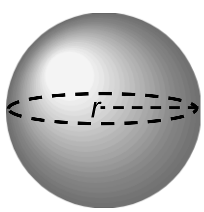
Circle area
$A = \pi r^2$

Circumference
$C = 2\pi r$



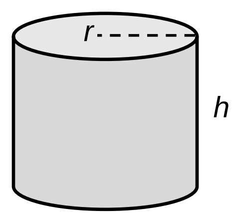
Sphere volume
$V = \frac{4}{3} \pi r^3$

Surface area
$A = 4\pi r^2$

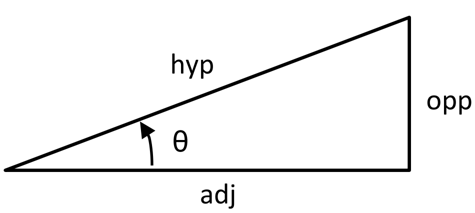


Cylinder volume
$V = \pi r^2 h$

Curved surface area
$A = 2\pi rh$



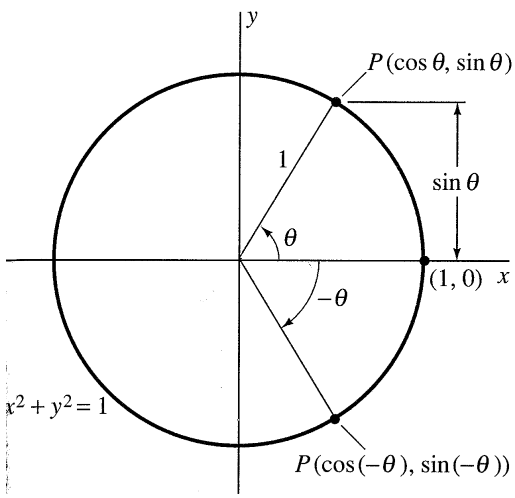
IV. Trigonometry



$\sin\theta = \frac{opp}{hyp}$ $\cos\theta = \frac{adj}{hyp}$ $\tan\theta = \frac{opp}{adj}$

$\tan x = \frac{\sin x}{\cos x}$

Rules on trigonometric functions can often be derived from the unit circle:



Such as:
$\sin(-\theta) = -\sin\theta$
$\cos(-\theta) = \cos\theta$
$\sin(\theta + \pi) = -\sin\theta$
$\cos(\theta + \pi) = -\cos\theta$
$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$
$\cos\left(\theta + \frac{\pi}{2}\right) = -\sin\theta$
$\sin(\theta + 2\pi) = \sin\theta$
$\cos(\theta + 2\pi) = \cos\theta$
$\sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$
$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$

Sum formulas:

$cos^2\theta + sin^2\theta = 1$

$\cos(a + b) = \cos a \times \cos b - \sin a \times \sin b$

$\cos(a - b) = \cos a \times \cos b + \sin a \times \sin b$

$\sin(a + b) = \sin a \times \cos b + \cos a \times \sin b$

$\sin(a - b) = \sin a \times \cos b - \cos a \times \sin b$

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \times \tan b}$$

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \times \tan b}$$

$\cos 2a = cos^2 a - sin^2 a = 2cos^2 a - 1 = 1 - 2sin^2 a$

$\sin 2a = 2 \sin a \times \cos a$

$cos^2 a = \frac{1}{2}(1 + \cos 2a)$; $sin^2 a = \frac{1}{2}(1 - \cos 2a)$

Transformation formulas:

$$\cos a \times \cos b = \frac{1}{2}[\cos(a + b) + \cos(a - b)]$$

$$\sin a \times \sin b = \frac{1}{2}[\cos(a - b) - \cos(a + b)]$$

$$\sin a \times \cos b = \frac{1}{2}[\sin(a + b) + \sin(a - b)]$$

$$\cos p + \cos q = 2\cos\frac{p + q}{2}\cos\frac{p - q}{2}$$

$$\cos p - \cos q = -2\sin\frac{p + q}{2}\sin\frac{p - q}{2}$$

$$\sin p + \sin q = 2\sin\frac{p + q}{2}\cos\frac{p - q}{2}$$

$$\sin p - \sin q = 2\sin\frac{p - q}{2}\cos\frac{p + q}{2}$$

For a triangle with edges *a*, *b*, *c* with respective opposite angles α, β, γ:
Law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\gamma$
Law of sines: $\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$

Inverse:
secant: $\sec x = \frac{1}{\cos x}$ cosecant: $\csc x = \frac{1}{\sin x}$ cotangent: $\cot x = \frac{1}{\tan x}$

Resolution:
if $\cos x = a$ then $x = \alpha + 2k\pi \vee x = -\alpha + 2k\pi$, for integer k and $\alpha = \text{acos}\, a$ in $[-\pi, \pi]$
if $\sin x = a$ then $x = \alpha + 2k\pi \vee x = \pi - \alpha + 2k\pi$, for integer k and $\alpha = \text{asin}\, a$ in $[-\pi, \pi]$

Values to know:

radian *degree*	0 0	$\pi/6$ 30	$\pi/4$ 45	$\pi/3$ 60	$\pi/2$ 90	π 180
sin	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	0
cos	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	-1
tan	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞	0

Hyperbolic functions:

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

V. Algebraic properties of usual functions

V. A – Roots

$$\sqrt[n]{x^m} = (\sqrt[n]{x})^m = x^{m/n}$$

V. B – Logarithms

$y = {}^b\log x \Leftrightarrow x = b^y$ (and hence $b^{{}^b\log x} = x$)
$\log 1 = 0$; ${}^b\log b = 1$
$\log(xy) = \log(x) + \log(y)$; $\log(x/y) = \log(x) - \log(y)$ (and hence $\log(1/y) = -\log y$)
$\log(x^a) = a\log x$

$${}^b\log x = \frac{{}^c\log x}{{}^c\log b} \quad (e.g.\, \log x = \frac{\ln x}{\ln 10} \Leftrightarrow {}^{10}\log x = \frac{{}^e\log x}{{}^e\log 10})$$

V. C – Exponents

$x^0 = 1$; $(xy)^r = x^r y^r$; $x^r x^s = x^{r+s}$; $\frac{x^r}{x^s} = x^{r-s}$; $(x^r)^s = x^{rs}$; $x^{-r} = \frac{1}{x^r}$
If $n \in \mathbb{N}^*, x \geq 0, y \geq 0$: $y = \sqrt[n]{x} \Leftrightarrow x = y^n$

VI. Limits

$\lim_{x\to+\infty} \log x = +\infty$ $\lim_{x\to 0} \log x = -\infty$

$\lim_{x\to+\infty} e^x = +\infty$ $\lim_{x\to-\infty} e^x = 0$

if $r > 0, \lim_{x\to 0} x^r = 0$; if $r < 0, \lim_{x\to 0} x^r = +\infty$

if $r > 0, \lim_{x\to+\infty} x^r = +\infty$; if $r < 0, \lim_{x\to+\infty} x^r = 0$

$\lim_{x\to 0} \frac{\log(1+x)}{x} = 1$ $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$ $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$\lim_{x\to+\infty} \frac{e^x}{x} = +\infty$ $\lim_{x\to-\infty} xe^x = 0$ $\lim_{x\to+\infty} \frac{\log x}{x} = 0$

if $r > 0, \lim_{x\to+\infty} \frac{e^x}{x^r} = +\infty$ if $r > 0, \lim_{x\to+\infty} x^r e^{-x} = 0$ if $r > 0, \lim_{x\to+\infty} \frac{\log x}{x^r} = 0$

VII. Differentiation

VII. A - Functions

$\boldsymbol{f(x)}$	$\boldsymbol{f'(x)}$
c	0
x	1
x^n	nx^{n-1}
c^x	$c^x \ln c$
$1/x$	$-1/x^2$
$1/x^n$	$-n/x^{n+1}$
$\sqrt{x}$	$1/(2\sqrt{x})$
$\ln x$	$1/x$
${}^c\log x$	$({}^c\log e)/x$
e^x	e^x
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
$\tan x$	$1/\cos^2 x$
$\arcsin x$	$1/\sqrt{1-x^2}$
$\arccos x$	$-1/\sqrt{1-x^2}$
$\arctan x$	$1/(1+x^2)$

VII. B - Operations

$(u+v)' = u' + v'$

$(cu)' = cu'$

$(c^u)' = c^u \ln c \; u'$

$(uv)' = u'v + uv'$

$(1/u)' = -u'/u^2$

$(u/v)' = (u'v - uv')/v^2$

$(v(u))' = v'(u)\, u'$

$(e^u)' = e^u u'$

$(\ln u)' = u'/u$

$(u^\alpha)' = \alpha u^{\alpha-1} u'$

$(\sin u)' = \cos(u)\, u'$

$(\cos u)' = -\sin(u)\, u'$

VIII. Integrals

VIII. A – Fundamental formulas

If F is a primitive of f, then $\int_a^b f(t)\,dt = F(b) - F(a)$

If $g(x) = \int_a^x f(t)\,dt$, then $g'(x) = f(x)$

$f(x) - f(a) = \int_a^x f'(t)\,dt$

VIII. B – Chasles' formulas

$$\int_a^c f(t)\,dt = \int_a^b f(t)\,dt + \int_b^c f(t)\,dt$$

$$\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$$

VIII. C – Positivity

If $a \leq b$ and $f \geq 0$, then $\int_a^b f(t)\,dt \geq 0$

VIII. D – Linearity

$$\int_a^b \alpha f(t) + \beta g(t)\,dt = \alpha \int_a^b f(t)\,dt + \beta \int_a^b g(t)\,dt$$

VIII. E – Inequality integration

If $a \leq b$ and $f \leq g$, then $\int_a^b f(t)\,dt \leq \int_a^b g(t)\,dt$

If $a \leq b$ and $m \leq f \leq M$, then $m(b-a) \leq \int_a^b f(t)\,dt \leq M(b-a)$

VIII. F – Partial integration

$$\int_a^b u(t)v'(t)\,dt = [u(t)v(t)]_a^b - \int_a^b u'(t)v(t)\,dt$$

IX. Function Analysis

- Find zero crossings of the function *(or component functions)*
- Find zero crossings of the derivative and sign around them *(and give tangent vectors)*
- Look at singularities

- Look at behavior at domain ends
- Draw likely points and function

X. Multivariable calculus

$$\nabla f(x_1, x_2, \dots, x_n) = \begin{pmatrix} f_{x_1} \\ f_{x_2} \\ \vdots \\ f_{x_n} \end{pmatrix}$$

The directional derivative of *f* in the direction of row vector *u*: $f_u = \frac{1}{\|u\|}(u \cdot \nabla f)$

A stationary point *p* is defined by $\nabla f(p) = 0$. It is

- a maximum if $|H| > 0$ and $\frac{\partial^2 f}{\partial x_1^2} < 0$
- a minimum if $|H| > 0$ and $\frac{\partial^2 f}{\partial x_1^2} > 0$
- a saddle-point if $|H| < 0$

where the Hessian *H* is the matrix of second-order derivatives of *f*

XI. Numeric methods

XI. A – Lagrange interpolation

The polynomial $p(x) = p_0(x) + p_1(x) + \dots + p_n(x)$ with

$$p_j(x) = y_j \prod_{\substack{k=0 \\ k \neq j}}^{n} \frac{x - x_k}{x_j - x_k}$$

interpolates the $n + 1$ support points $\{(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)\}$.

XI. B – Regula Fasli

The intersection point of the line $(x_0\ x_1)$ and the x-axis is *(c,0)* where $c = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$.

If $f(c) = 0 \pm \varepsilon$, the root is found. If $f(c)f(x_0) < 0$, the root is to the left of *c*. If $f(c)f(x_0) > 0$, the root is to the right of *c*.

XI. C – Picard iteration

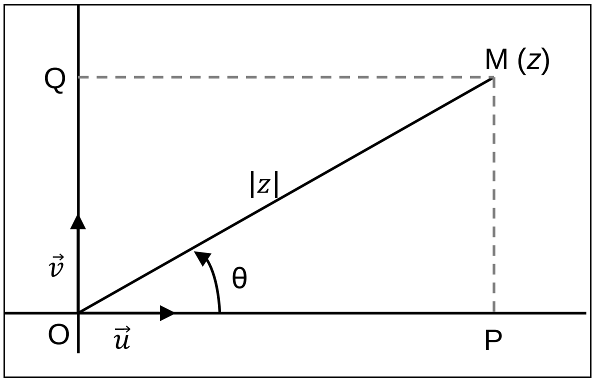
Change the equation $f(x) = 0$ into $g(x) = x$. Approximation of the root is given by $x_{n+1} = g(x_n)$ with a suitable initial value x_0.

XI. D – Newton-Raphson iteration

A solution to $f(x) = 0$ may be found by iterating $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ with a suitable initial value x_0.

XII. Complex calculus

Algebraic form: $z = x + iy$
Polar form: $z = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}\ ,\ |z| > 0$



$\overrightarrow{OM} = x\vec{u} + y\vec{v}$
$\overline{OP} = x = Re(z) = |z|\cos\theta$
$\overline{OQ} = y = Im(z) = |z|\sin\theta$
$OM = |z| = \sqrt{x^2 + y^2}$

Algebraic operations:
$z + z' = (x + iy) + (x' + iy') = (x + x') + i(y + y')$
$zz' = (x + iy)(x' + iy') = (xx' - yy') + i(xy' + x'y)$

Conjugate:
$z = x + iy = |z|e^{i\theta}$; $\bar{z} = x - iy = |z|e^{-i\theta}$
$x = \frac{1}{2}(z + \bar{z})$; $y = \frac{1}{2i}(z - \bar{z})$
$\overline{z + z'} = \bar{z} + \overline{z'}$; $\overline{zz'} = \bar{z}\overline{z'}$
$z\bar{z} = x^2 + y^2 = |z|^2$
$\frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{x^2 + y^2} + i\frac{-y}{x^2 + y^2} = \frac{1}{|z|}e^{-i\theta}$

Product and ratio:
$zz' = \left(|z|e^{i\theta}\right)\left(|z|'e^{i\theta'}\right) = |z||z|'e^{i(\theta+\theta')}$; $|zz'| = |z||z'|$
$\frac{z}{z'} = \frac{|z|e^{i\theta}}{|z|'e^{i\theta'}} = \frac{|z|}{|z|'}e^{i(\theta-\theta')}$; $\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}$
$z^n = \left(|z|e^{i\theta}\right)^n = |z|^n e^{in\theta}$

Triangle inequality:
$\big||z| - |z'|\big| \le |z + z'| \le |z| + |z'|$

Euler's formulas:
$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$; $\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$

De Moivre's formula:
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)\ \ i.e.\ \left(e^{i\theta}\right)^n = e^{in\theta}$